that we can choose the directions of the infinitely distant points P_1 and P_2 (in an infinite variety of ways) so as to make the expression $\lambda = \tan^{-1}(\beta/\alpha)$ for the astronomical longitude have any value we wish.[1]

From the form of the function W, it is evident that the level surfaces (1) are, in general, *not* surfaces of revolution. If, in particular, the distribution of the earth's matter were such that the function U assumed the form $f(v, z)$, where $v = \sqrt{x^2 + y^2}$, then the level surfaces would be surfaces of revolution, the east-and-west curves would be identical with the curves of constant astronomical latitude, and the curves of constant astronomical longitude would be the meridian curves[2] of these surfaces of revolution.

Hence Theorem II yields the following corollary.

COROLLARY. *The locus of the brilliant points, with respect to an infinitely distant source P_1 and an infinitely distant recipient P_2, of the one-parameter family of curves which are cut from a surface of revolution by planes perpendicular to the axis, is a meridian curve of this surface of revolution.*[3]

The photograph on page 73 shows two such curves (due to two sources of light) on a brass sphere which has been properly scratched with emery cloth while rotating in a lathe. In reality, the sources and the recipient (observer's eye) are not infinitely distant, but they are practically so.

A THEOREM IN THE MODERN PLANE GEOMETRY OF THE ABRIDGED NOTATION.

BY ROBERT E. BRUCE, Boston University.

NOTE BY THE EDITORS. The following paper is another contribution fulfilling the spirit of the editorial in the January issue. While the subject matter is strictly elementary, the methods are elegant and forceful and should furnish an inspirational study for those interested in modern geometry. Those who may wish to read an introduction to the abridged notation will find chapter IV of Salmon's *Conic Sections* helpful.

Introduction. In common with polar reciprocation, projection, and certain other methods of modern geometry, the abridged notation may be used either for the proof of a theorem presupposed to be true or for the discovery of new theorems the exact form of which is unsuspected until the proof is complete. For either purpose an identity in terms of the abridged notation is selected and any number of algebraic operations performed upon it leading to a final identity. The first identity corresponds to the hypothesis, the last to the conclusion, and the intervening transformations to the proof. If the form of the theorem is

[1] It is also evident that the locus of the billiant points, with respect to a source P_1 and a recipient P_2, of any one-parameter family of curves on a surface σ, passes through the brilliant points, with respect to P_1 and P_2, of σ.

[2] A meridian curve of a surface of revolution is a curve which is cut from this surface by a plane which passes through the axis of rotation.

[3] If a surface of revolution were scratched along its meridian curves, the locus of the brilliant points would be a more complex curve. Such a locus has been considered by the author for the case of a sphere, in Vol XX, No. 10, p. 299 of this MONTHLY.

already suspected the identity of the hypothesis and the transformations are, as a matter of course, so selected as to bring the desired result. If, on the other hand, one desires to discover some theorem for the satisfaction of the discovery rather than for the theorem itself, the identity of hypothesis and the transformations may be selected almost at random; the work when done being interpreted in whatever geometry one chooses. In the following, the main theorem illustrates this latter use of the notation while the lemma proved in connection with corollary 8 illustrates the former.

Proof of Main Theorem. Let the k's be constants other than zero and let the S's be expressions of the same degree in the variables.

If

$$k_1S_1 + k_2S_2 \equiv k_3S_3 + k_4S_4$$

then

$$k_1S_1 - k_3S_3 \equiv k_4S_4 - k_2S_2$$

and

$$k_1S_1 - k_4S_4 \equiv k_3S_3 - k_2S_2.$$

Limiting the interpretation for simplicity to the case in point geometry where there are but two variables and where the S's are of the second degree we have the following:

If a conic through the intersections of conics S_1 and S_2 passes through the intersections of conics S_3 and S_4, then there is a conic through the intersections of S_1 and S_3 that passes through the intersections of S_4 and S_2. Similarly for S_1, S_4 and S_3, S_2. Hence the following:

Theorem. *If it is possible to divide four given conics into two groups of two each in such a way that the eight points of intersection within the groups are on a conic, then there will be a conic through the eight points of intersection in each of the other arrangements of the conics into two groups of two each.*

It is obvious that various other theorems may be derived from the proof by changing the degree of S, the number of variables, or the particular geometry used.

The following are some of the corollaries of the theorem as stated:

Corollary 1. Consider three intersecting conics with their nine pairs of opposite chords of intersection. Denote the conics by S_1, S_2, and S_3 respectively. Denote the pairs of opposite chords of intersection as follows: One pair for S_1 and S_2 by C_{12}', a second pair by C_{12}'', and the third by C_{12}'''. So for S_1, S_2 and S_2, S_3. Any one of these pairs of chords is a degenerate conic. Consider the four conics S_1, C_{13}', S_2, and C_{23}' grouping them as follows: (S_1, C_{13}'); (S_2, C_{23}'). The eight intersections within the groups are all on the conic S_3. Hence there exist two eight-point[1] conics,—one through the intersections of S_1 with S_2 and of C_{13}' with C_{23}', and the other through the intersections of S_1 with C_{23}', and of S_2 with C_{13}'. But the group[2] (S_1, C_{13}') may, aside from the above, be associated severally

[1] For sake of brevity the phrase "eight-point conics" will be used to denote the conics involved in the conclusion of the main theorem.

[2] This word is of course used in a non-technical sense.

with the groups (S_2, C_{23}'') and (S_2, C_{23}'''), the hypothesis of the theorem being fulfilled in each case. Hence for the group (S_1, C_{13}') there exist six eight-point conics derived through its association with S_2 and the various pairs of opposite chords of intersection of S_2 and S_3. But the group (S_1, C_{13}') may also be associated with the group (C_{23}', C_{23}''), since these two degenerate conics intersect on S_3, thus giving two more eight-point conics. Similarly for the groups (C_{23}', C_{23}''') and (C_{23}'', C_{23}'''). Hence there exist 12 eight-point conics for the various groupings above, using (S_1, C_{13}') as one pair of conics. But in place of (S_1, C_{13}') there may be substituted any one of the following pairs of conics: (S_1, C_{13}''), (S_1, C_{13}'''), (C_{13}', C_{13}''), (C_{13}', C_{13}'''), (C_{13}'', C_{13}'''). Each substitution gives, of course, 12 new eight-point conics by taking severally the various groups we have used above with (S_1, C_{13}'). Hence there exist 72 eight-point conics for the cases considered. But just as S_1 and S_2 with the chords used intersect upon S_3, so also do S_1 and S_3 with their chords intersect upon S_2, and S_2 and S_3 upon S_1. Hence there exist 216 eight-point conics through the various intersections. The above may be summarized as follows:

Given any three intersecting conics with their chords of intersection; there exist 216 *different conics each containing eight of the points of intersection of the conics and chords with one another.*

This corollary has been worked out in detail for the sake of clearness. A moment's consideration, however, makes it plain that the result may be found at once from the following product: $2 \times {}_4C_2 \times {}_4C_2 \times 3 = 216$.[1] It is also evident that the same method may be applied to the cases where the hypothesis gives four or more intersecting conics with their chords of intersection.

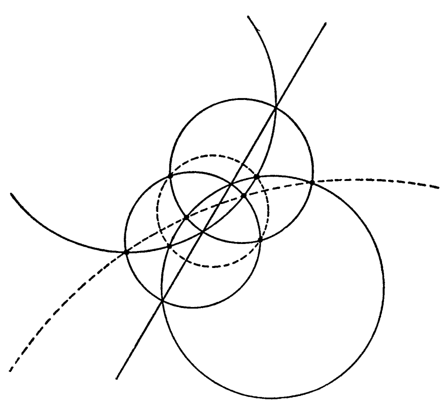

FIG. 1.

Corollary 2. If two pairs of conics have segments of the same two straight lines as opposite chords of intersection, the sixteen other points of intersection of the conics with each other are in two groups of eight each on two additional conics.

The two chords form a degenerate conic. And by hypothesis the four conics

[1] Where ${}_nC_r$ is the number of combinations of n things taken r at a time.

intersect upon it in two groups of two each. The conclusion, then, follows at once from the main theorem. The next two corollaries are special cases of the above.

Corollary 3. If two pairs of circles have different segments of the same line as their common chords, then the eight other points of intersection of the circles with each other are in groups of four each on two other circles. (See Fig. 1.)

The line at infinity is the second chord satisfying the hypothesis of corollary 2. The missing intersections are the circular points at infinity. Hence the conics of the conclusion are circles.

Corollary 4. Consider two quadrilaterals with different segments of the same two lines as diagonals. (See Fig. 2.) The pairs of opposite sides of the quadrilaterals are degenerate conics. Thus the hypothesis of corollary 2 is fulfilled. Hence there are two eight-point conics through the sixteen points of intersection of the sides of one quadrilateral with the sides of the other. This corollary is the basis of the construction of the following corollary.

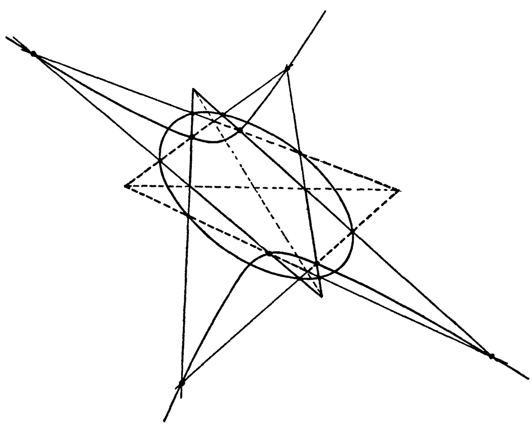

FIG. 2.

Corollary 5. *Construction of the conic through five given points.*[1]

Number the points 1, 2, 3, 4, 5. Let line 24 be α', 35 be β', 25 be b', and 34 be a'. Draw any two convenient lines, a and β, through 1.

Let Y be the intersection of a and a', W of a and b', I of β and α', and III of β and β'. Let the line WI be d, and $Y\,III$ be d'.

Let X be the intersection of d and a', II of d and β', IV of d' and α', and Z of d' and b'.

Let line XZ be b, and $II\,IV$ be α.

Let 6 be the intersection of b and β, 7 of a and α, 8 of b and α.

The points 6, 7, 8 are on the conic through 1, 2, 3, 4, 5. Other sets of points may be found by varying the arbitrary lines a and β.

[1] No diagram is drawn because the construction will be better appreciated by the reader if he draws the figure step by step. A figure is given in connection with a briefer statement of the construction added below.

Proof of the construction. The conic $\alpha\beta$ intersects the conic $\alpha'\beta'$ on the conic dd'. The conic ab intersects the conic $a'b'$ on the conic dd'. Hence the intersections of ab with $\alpha\beta$ and of $a'b'$ with $\alpha'\beta'$ are on a conic. But these points of intersection are the points 1, 2, 3, 4, 5, 6, 7, 8. Hence 6, 7, and 8 are on the conic through 1, 2, 3, 4, 5.

The following briefer statement of the method of construction has been found to satisfy the proof in all cases tested. (See Fig. 3.)

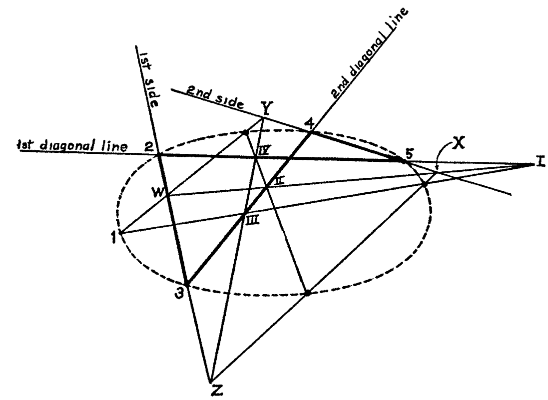

FIG. 3.

Selecting four of the given points, draw either pair of opposite sides and the two diagonals of the quadrilateral[1] of which these points are the vertices. Draw any two convenient lines through the fifth point.[2] Let the first of these lines intersect the first[3] and second diagonals in points *I* and *III* respectively. And let the second line intersect the first and second sides in points *W* and *Y* respectively. Call the points where the line *Y III* meets the first side and the first diagonal *Z* and *IV* respectively; and the points where the line *W I* meets the second side and the second diagonal *X* and *II* respectively. The following intersections are on the conic: *I III* with *X Z*, *X Z* with *II IV*, *II IV* with *Y W*. By varying the lines through the fifth point as many points as desired may be found.

Corollary 6. *Pascal's Theorem:* "The intersections of the opposite sides of a hexagon inscribed in a conic are collinear." (See Figure 4.)

Let $ABCDEF$ be the given hexagon. Draw $C'F'$ any line that meets the conic, and form the two inscribed quadrilaterals $ABCF$ and $C'DEF'$. The degenerate conic formed by the lines BC and AF intersects the conic of the lines AB and CF on the given conic. Similarly for conics $C'D$, EF' and $C'F'$, ED. Hence these four conics obey the hypothesis of the main theorem and by pairing

[1] That is, the ordinary convex quadrilateral of the elementary euclidean geometry,—not the complete quadrilateral. If no three of the points are in the same straight line a selection of this character is always possible.

[2] In the figure this point is 1.

[3] The numbering throughout is arbitrary.

the conics as follows: $BC \cdot AF$ with $EF' \cdot C'D$ and $AB \cdot CF$ with $C'F' \cdot ED$ the points 1, 2, 3, 4, 5, 6, 7, 8 are seen to be on a conic. But if the line $C'F'$ coincides with the line CF, the points 6, 7, 1, 2, 5 are collinear on the line CF. Hence the

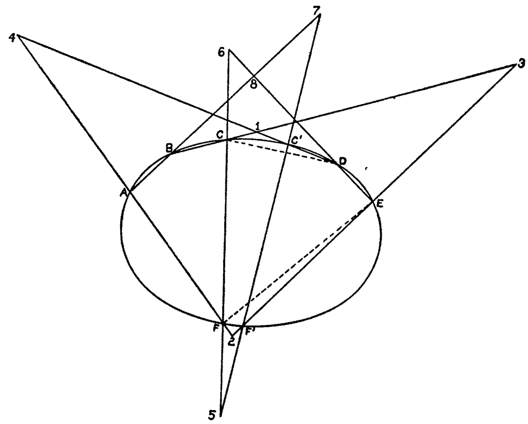

FIG. 4.

eight-point conic is degenerate and the points 4, 8, 3 are collinear on a line other than CF. But these points are the intersections of the opposite sides of the hexagon.

Corollary 7. "If three conics have a common chord the opposite common chords are concurrent." (*Salmon's Conic Sections*, Art. 266.) See Figure 5.

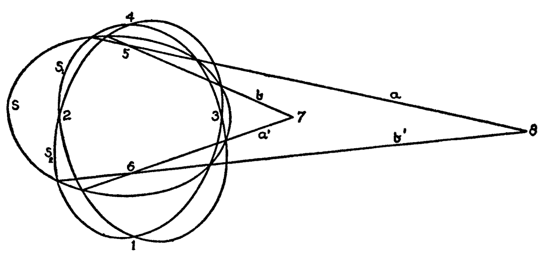

FIG. 5.

If the chords a and a' are associated with S_1, and b and b' with S_2, the two pairs of conics thus obtained obey the hypothesis of the main theorem, since the intersections within the groups are on S. Hence by associating chords $a \cdot a'$ with chords $b \cdot b'$ and conic S_1 with conic S_2 the points 1, 2, 3, 4, 5, 6, 7, 8 are seen to be on a conic. If b coincides with a the three conics have a common chord and the points 2, 3, 5, 7, 8 are collinear upon it. Hence the eight-point conic is degenerate and the points 4, 6, 1 are collinear. 6 is thus the point of concurrency of the three opposite chords of intersection. It is obvious that the proof may be applied to any three conics having a common chord.

Corollary 8. "If two conics have double contact with a third, their chords of contact with the third and a pair of their chords of intersection with each other are concurrent." (*Salmon's Conic Sections*, Art. 263.)

For the proof of this and other corollaries the following lemma is used in addition to the main theorem.

Lemma. If a conic intersects itself or is tangent to itself, it is degenerate and consists of two straight lines through the points of intersection or tangency.

If $S = 0$ is the equation of a conic, and $\alpha = 0$ and $\beta = 0$ the equations of straight lines, and k a constant other than zero; then $S + k\alpha\beta = 0$ may be used to represent any conic intersecting S or tangent to S, the particular relationship depending on the relationship of α and β to S. (*Salmon's Conic Sections*, Arts. 249, 251.)

Since by hypothesis S is to intersect itself or to be tangent to itself, $S \equiv m(S + k\alpha\beta)$, where m is a constant other than zero.

Hence

$$S \equiv \frac{mk}{1 - m}\alpha\beta.$$

Now $m \neq 1$ since otherwise, from the first identity, the product $\alpha\beta$ would be identically zero. Hence the locus S is the same as the straight lines α and β combined.

Proof of corollary 8. Let S_1 have double contact with S and let S_2 be tangent to S at G and intersect it at H and K. The following groups of conics fulfill the hypothesis of the main theorem: (S_1, $EF \cdot EF$) and (S_2, $GH \cdot GK$).

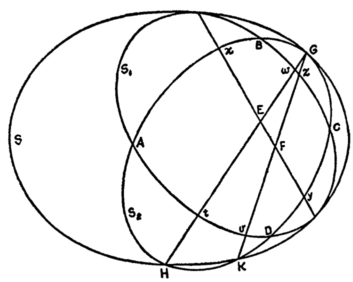

FIG. 6.

Hence there is a conic through the points $ABCD$ which is tangent to the line HG at E and to the line KG at F. But if F and E coincide, this conic is tangent to itself. Hence it is degenerate consisting of the two lines AC and BD passing through the point EF. Moreover, under these conditions the conic S_2 has double contact with S. Apparently any conic having double contact with S may be derived in this way and hence the corollary is proved.

Corollary 9. The second eight-point conic in figure 6 passes through the points t, v, w, z and has double contact with S_2 at x and y. Moreover, if S_2 has double contact with S, w coincides with z and t coincides with v, and this second eight-point conic has double contact with S_1 also. Hence the following: If two conics have double contact with a third conic they also have double contact with a fourth conic, the chords of contact being different segments of the same two lines.

Corollary 10. If a pair of opposite chords of intersection of each of two conics with a third meet at the same point, that is, if the four lines are concurrent, a pair of opposite chords of intersection of the two conics with each other pass through the point.

Grouping each of the two conics with its given chords of intersection with the third conic, we have four conics obeying the hypothesis of the main theorem. Hence there is an eight-point conic through the intersections of the two conics with each other and the intersections of the chords with each other. But since the four points of intersection of the chords with each other are coincident the eight-point conic either intersects itself or is tangent to itself. Hence it is degenerate and the corollary is established.

It is apparent that corollary 8 is a special case of the above.

Corollary 11. If in corollary 2 the two straight lines are coincident we have the following: If different segments of the same straight line are chords of contact of two pairs of conics having double contact, then the sixteen points of intersection of the two pairs of conics with each other are in two groups of eight each on two additional conics.

Several special cases of this corollary follow.

Corollary 12. The four points of intersections of any two hyperbolas with each other are on a conic through the four points of intersection of their asymptotes with each other.

This follows directly from corollary 11 and from the fact that any hyperbola and its asymptotes have double contact, the chord of contact being the line at infinity. The other eight-point conic in this case gives the following:

Corollary 13. The four points of intersection of one hyperbola with the asymptotes of a second are on a conic through the four points of intersection of the second hyperbola with the asymptotes of the first.

Corollary 14. Given two pairs of concentric circles; the eight points of intersection of the circles with each other are in two groups of four each on two additional circles.

This follows from the fact that concentric circles have double contact on the line at infinity. Moreover the eight missing points of intersection are the circular points at infinity and thus the conics of the conclusion are circles.

Corollary 15. It is obvious that a hyperbola with its asymptotes and two concentric circles may also be used to discover two eight-point conics as in corollaries 12 to 14.

Corollary 16. Given a tangent to a parabola and two conics having double contact, the chord of contact being a segment of that line parallel to the axis of the parabola which passes through the point of contact of the tangent and parabola; the intersections of the parabola with either conic and of the tangent with the other are on a conic. The species of this conic is the same as the species of the given conic which meets it on the tangent. If this conic is a circle so is the resulting conic.

This corollary is seen to be true from the following considerations: The para-

bola has double contact with the tangent and the line at infinity, the parallel to the axis being the chord of contact. Hence corollary 11 may be applied to this case. The nature of the points of intersection of the given conics with the line at infinity fixes the species of the resulting conics since these points are on the resulting conics.

BOOK REVIEWS.

UNDER THE DIRECTION OF W. H. BUSSEY.

The Teaching of Arithmetic. By DAVID EUGENE SMITH. Ginn and Company, Boston, 1913. v+196 pages.

A Textbook on the Teaching of Arithmetic. By ALVA WALKER STAMPER. American Book Company, New York, 1913. 284 pages.

Considerable attention should be paid to the teaching of arithmetic by all of those who have in charge the training of teachers and administrative officers of the public school system. Hence the universities as well as the normal and training schools should provide instruction along these lines. Both of these texts under discussion are well adapted for this purpose and it is to be hoped that they will enjoy wide use.

Eleven years ago Professor Smith and Professor McMurry of Teachers College prepared for the Teachers College Record an able article on the teaching of arithmetic. This was well adapted for instruction purposes and was so used until the edition was exhausted. In 1909 Professor Smith prepared for the same journal a new article with the same title and used some of the same material. This was also published in book form by Teachers College. The latter has been somewhat revised and expanded for this work from the press of Ginn and Co. The discussion of number games, briefly treated before, has been elaborated into a useful chapter. The chapter entitled "Subjects for Experiment" deserves the close attention of educators. Similar suggestions have been made in the MONTHLY for such experiments in connection with the high school and college mathematics. In this treatment some reference should certainly have been made to the tests for measurement of the efficiency of instruction in arithmetic, notably those suggested by S. A. Courtis. As a part of the recent examination of the New York City Schools such tests were used and in schools all over the country similar tests are being made. A discussion by Professor Smith of the value of these tests would have been a valuable and timely addition to the work.

The second of these works, by A. W. Stamper, takes up the topics much more in detail and does not treat the larger and more general phases of the subject. For this reason this book may be found more useful by normal schools and training classes. Some of the historical matter in this text is misleading. Thus it is quite incorrect to say that "the science of algebra originated in the fourth century B. C. in the Greek city of Alexandria in Egypt." Sciences do not originate in this definite and precise way and Alexandria is not connected with its origin